

Aspects of affine Toda field theory on a half line

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Abstract

The question of the integrability of real-coupling affine toda field theory on a half line is discussed. It is shown, by examining low-spin conserved charges, that the boundary conditions preserving integrability are strongly constrained. In particular, among the cases treated so far, $e_6^{(1)}$, $d_n^{(1)}$ and $a_n^{(1)}$, $n \geq 2$, there can be no free parameters introduced by such boundary conditions; indeed the only remaining freedom (apart from choosing the simple condition $\partial_1 \phi = 0$), resides in a choice of signs. For a special case of the boundary condition, accessible only for $a_n^{(1)}$, it is pointed out that the classical boundary bound state spectrum may be related to a set of reflection factors in the quantum field theory. Some preliminary calculations are reported for other boundary conditions, demonstrating that the classical scattering data satisfies the weak coupling limit of the reflection bootstrap equation.

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1. Introduction

Over the last few years, there has been some progress in understanding the particle scattering of real coupling affine Toda field theory [1-6]. Until recently, the theory has been formulated on the full line $|x^1| \leq \infty$. However, it might be interesting for several reasons to consider the theory defined on a half line, or on an interval, in which case it becomes important to consider carefully the effect of adding boundary conditions at one, or two, points. Consider, for example, the restriction to a half line. Affine Toda field theory is a lagrangian field theory and the addition of a boundary (at $x^1 = 0$) requires a lagrangian of the form

$$\bar{\mathcal{L}} = \theta(-x^1)\mathcal{L} - \delta(x^1)\mathcal{B}, \quad (1.1)$$

where \mathcal{B} , which is taken to be a functional of the fields but not their derivatives, represents the boundary condition. In other words, at the boundary $x^1 = 0$

$$\frac{\partial\phi}{\partial x^1} = -\frac{\partial\mathcal{B}}{\partial\phi}. \quad (1.2)$$

The first term in (1.1) contains the usual affine Toda lagrangian [7]:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - V(\phi) \quad (1.3)$$

where

$$V(\phi) = \frac{m^2}{\beta^2} \sum_0^r n_i e^{\beta\alpha_i \cdot \phi}. \quad (1.4)$$

In (1.4), m and β are real constants, α_i , $i = 1, \dots, r$ are the simple roots of some Lie algebra g , and $\alpha_0 = -\sum_1^r n_i \alpha_i$ is an integer linear combination of the simple roots; it corresponds to the extra spot on an extended (untwisted or twisted) Dynkin-Kac diagram for \hat{g} . The coefficient n_0 is taken to be one.

Affine Toda field theory on the full line is classically integrable [7,8]; there is a Lax pair representation of the field equations and, as a consequence of this, there are infinitely many conserved charges in involution. The charges Q_s are labelled by spins s which take values equal to the exponents of the Lie algebra g modulo its Coxeter element. On the half line, translation invariance is obviously lost and the best one can hope for by way of conserved quantities is some modification of the parity even combination $Q_s + Q_{-s}$. For example, the energy is $\hat{Q}_1 + \hat{Q}_{-1} + \mathcal{B}$, (hatted quantities are integrated densities on the half line), and is conserved whatever the form of the boundary condition might be.

Some time ago, Sklyanin and others [9], and, more recently, Ghoshal and Zamolodchikov [10,11] considered the question of the sine-Gordon theory with a boundary of exactly this type. The conclusion finally arrived at seems to be that the most general boundary condition permitted by classical considerations [12] has the form:

$$\frac{\partial\phi}{\partial x^1} = \frac{a}{\beta} \sin \beta \left(\frac{\phi - \phi_0}{2} \right) \quad \text{at} \quad x^1 = 0, \quad (1.5)$$

where a and ϕ_0 are arbitrary constants, and β is the sine-Gordon coupling constant. Clearly, to preserve integrability, the form of the boundary term in the lagrangian is strongly constrained.

In affine Toda field theory, it was shown recently [13], for the case $g = a_n$, that the condition at the boundary is generally even more restricted than it is for the sine-Gordon situation; indeed, there is only a discrete ambiguity in the choice of \mathcal{B} . It was found that the general form of the boundary condition must be

$$\mathcal{B} = \frac{m}{\beta^2} \sum_0^r A_i e^{\frac{\beta}{2} \alpha_i \cdot \phi}, \quad (1.6)$$

where the coefficients A_i , $i = 0, \dots, r$ are a set of real numbers. Moreover, for $n > 1$ there was found to be an extra constraint on the coefficients, namely:

$$\textbf{either } |A_i| = 2, \text{ for } i = 0, \dots, n \textbf{ or } A_i = 0 \text{ for } i = 0, \dots, n. \quad (1.7)$$

Subsequent investigation has led to a general statement expected to be applicable to all simply-laced affine Toda theories:

$$\textbf{either } |A_i| = 2\sqrt{n_i}, \text{ for } i = 0, \dots, r \textbf{ or } A_i = 0 \text{ for } i = 0, \dots, r. \quad (1.8)$$

The sine- and sinh-Gordon theories appear to be the only ones for which there is a continuum of possible boundary conditions; for the others, the only ambiguity at the boundary (up to an additive constant) is a choice of signs.

The argument leading to the necessity of (1.7) required only that spin 2 charges be preserved (in the sense that there is a boundary term Σ_2 for which $\hat{Q}_2 + \hat{Q}_{-2} - \Sigma_2$ is conserved). For (1.8), charges of higher spin must be invoked, while a complete argument requires a generalisation of the Lax pair idea taking into account the existence of the boundary.

Notice that for every choice of Lie algebra other than a_n , the boundary condition (1.2), together with the form of \mathcal{B} , fails to permit the vacuum solution $\phi = 0$, unless $\mathcal{B} = 0$. This fact alone indicates that in general the quantum field theory will be considerably complicated by the addition of the boundary term. However, it is also this fact which makes the theory with a boundary an interesting challenge.

As a final remark, notice that the permissible boundary conditions fall into several types in the following sense. The original affine Toda lagrangian is invariant under translations of the field by vectors of the lattice dual to the root lattice of the Lie algebra:

$$\phi \rightarrow \phi + \frac{2\pi i}{\beta} \lambda \quad \text{where } \lambda \cdot \alpha_i = \text{integer.} \quad (1.9)$$

The effect of such a transformation on the field theory with a boundary condition (1.6) is to relate different boundary conditions by altering the relative signs of the terms in \mathcal{B} . Clearly, the term depending on the extra affine root α_0 has a sign which is determined once the other sign changes have been made by using a suitable choice of the vector λ . For example, in the a_n theory boundary conditions which differ by an even number of sign changes are related by a transformation of the type (1.9); in this sense, there are just two types. While the transformation (1.9) does not preserve the reality of the Toda fields, it does relate different ground states in the complex version of the theory and is important for understanding the existence of the complex solitons [14].

2. Factorisable scattering with a boundary

From the point of view of the quantum field theory, the first question to ask is what might be the optimal scenario if the theory with a boundary remains integrable in the quantum domain. It is believed that affine Toda field theory on the full line, restricted to real coupling, is a relatively simple field theory with a spectrum of r scalar particles, whose scattering is purely elastic and factorisable. Conjectures have been made for their spectrum and S-matrices; these are consistent with the bootstrap idea and compatible with low order perturbation theory [1-4]. In fact, the theories split naturally into two classes: those based on the ‘self-dual’ affine Dynkin-Kac diagrams ($a^{(1)}, d^{(1)}, e^{(1)}$ and $a_{\text{even}}^{(2)}$), and the rest which fall into dual pairs [6]. For example, for the self-dual theories, the quantum spectrum of conserved charges (masses etc), and the quantum couplings are essentially identical to the corresponding classical data.

Following the old ideas of Cherednik [15], the best one could expect in the theory including a boundary is that the particle spectrum remains the same, and that far from the boundary the particles scatter as if the boundary were absent. The extra ingredient on the half line is that a particle approaching the boundary is scattered elastically from it, with its *in* and *out* states related by a reflection factor (and of course a reversal of momentum). Ie, in an integrable theory with distinguishable scalar particles it might be expected that

$$|a, -\theta_a \rangle_{\text{out}} = K_a(\theta_a) |a, \theta_a \rangle_{\text{in}}, \quad (2.1)$$

where a labels the particle, and θ_a is its rapidity. Moreover, if the scattering theory remains factorisable, then the scattering of any set of particles from the boundary can be computed using the set of one-particle reflection factors, and the set of two-particle S-matrices. The addition of the boundary may also influence the spectrum by introducing new kinds of states.

It must be remembered that it is one thing to say what might be expected in the best possible case, and quite another to establish it. It is quite possible that this scenario only pertains to a subset of the classical boundary conditions, or indeed to none of them; the question remains to be settled.

On the other hand, if it is supposed that the scattering with a boundary is factorisable in the manner described, then it is possible to generate a number of consistency relations, based on the bootstrap, which ought to be satisfied by the reflection factors K_a . These have been formulated and explored recently by Fring and Köberle [16], by Sasaki [17], and by Ghoshal and Zamolodchikov [10], the latter in a more general context. It has proved possible to conjecture a variety of solutions to the ‘reflection bootstrap equations’, some of which are described in these references. In [13] the question of finding solutions of the affine Toda bootstrap related to actual boundary conditions was addressed, and some suggestions were made concerning the spectrum of boundary bound states which might help to identify the reflection factors for a particular boundary condition among the plethora of solutions which have been found to the reflection bootstrap.

The relevant consistency conditions to be satisfied by the reflection factors are:

$$K_a^0(\theta_a) K_a^0(\theta_a - i\pi) = S_{aa}(2\theta_a), \quad (2.2)$$

and

$$K_c^0(\theta_c) = K_a^0(\theta_a) K_b^0(\theta_b) S_{ab}(\theta_a + \theta_b). \quad (2.3)$$

The second relation corresponds to the bulk coupling $ab \rightarrow c$ which appears as a forward channel bound state pole in the scattering of particles a and b , at a particular relative rapidity (for which $\theta_a = \theta_c - i\bar{\theta}_{ac}^b$, $\theta_b = \theta_c + i\bar{\theta}_{bc}^a$), corresponding to the total energy and momentum of a and b coinciding with that of particle c . The usual conventions have been adopted ($\bar{\theta} = \pi - \theta$). In eqs(2.2),(2.3) the superscript 0 refers to the ground state of the boundary.

When the bootstrap is applicable and, as here, the scattering is diagonal, the first relation (2.2) follows from repeated application of the second, eq(2.3). Note, though, for the theory corresponding to a_1 there are no relations of the second type.

It is interesting to remark that in a weak coupling limit, the particles become free far from the boundary and $S_{ab} \rightarrow 1$. However, particles must rebound from the boundary even if they are free far away from it; therefore the small coupling limit of the reflection factors need not be unity. Taking the limit as the coupling tends to zero for eq(2.3) reveals that the ‘classical’ coupling factors ought to satisfy a bootstrap of their own, namely

$$K(\theta_c) = K(\theta_a)K(\theta_b), \quad (2.4)$$

where the rapidities are related to each other as for (2.3). Examples will be given below which demonstrate that this remark is not a trivial one. Indeed, it corresponds to a phenomenon which may be of interest to anyone who has studied inverse scattering theory.

3. Higher spin charges

The spin even charges in a_n theories play an important rôle because they discriminate between mass degenerate conjugate particles. If they are not preserved on the half line, the boundary may allow the particles to mix with their conjugates on the rebound.

In the absence of a full Lax pair treatment of the half line problem, the pedestrian approach adopted in [13] will be employed to explain the necessity of the constraints (1.6) and (1.8). A fuller (and more satisfying) treatment will be described elsewhere [18]. For related discussions of the problem on the full line, see for example [19].

The spin ± 3 densities corresponding to the spin ± 2 charges for the whole line may be described by the general formulae (using light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$):

$$T_{\pm 3} = \frac{1}{3}A_{abc}\partial_\pm\phi_a\partial_\pm\phi_b\partial_\pm\phi_c + B_{ab}\partial_\pm^2\phi_a\partial_\pm\phi_b, \quad (3.1)$$

where the coefficients A_{abc} are completely symmetric and the coefficients B_{ab} are antisymmetric. For constructing conserved quantities, the densities must satisfy

$$\partial_{\mp} T_{\pm 3} = \partial_{\pm} \Theta_{\pm 1} \quad (3.2)$$

and explicit calculation reveals

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_{\pm} \phi_a V_b, \quad V_b = \frac{\partial V}{\partial \phi_b}, \quad (3.3)$$

with the constraint

$$A_{abc} V_a + B_{ab} V_{ac} + B_{ac} V_{ab} = 0. \quad (3.4)$$

Eq(3.4) implies

$$\frac{1}{\beta} A_{ijk} + B_{ij} C_{ik} + B_{ik} C_{ij} = 0, \quad (3.5)$$

where it is useful to define

$$A_{ijk} = A_{abc} (\alpha_i)_a (\alpha_j)_b (\alpha_k)_c, \quad B_{ij} = B_{ab} (\alpha_i)_a (\alpha_j)_b, \quad (3.6)$$

and

$$C_{ij} = \alpha_i \cdot \alpha_j.$$

Eq(3.5) implies that B_{ij} is very restricted: it is non-zero only for the $a_n^{(1)}$ cases (as expected since the exponent 2 does not occur elsewhere) and, in those cases, $B_{ij} = 0$ except for $j = i \pm 1 \bmod n + 1$, and $B_{i-1} i = B_{i i+1}$, $i = 1, \dots, n + 1$.

Rewriting the conditions (3.2) in terms of the variables x^0, x^1 ,

$$\partial_0 (T_{+3} - \Theta_{+1} \pm (T_{-3} - \Theta_{-1})) = \partial_1 (T_{+3} + \Theta_{+1} \mp (T_{-3} + \Theta_{-1})), \quad (3.7)$$

implies that the combination $(T_{+3} - \Theta_{+1} + T_{-3} - \Theta_{-1})$ is a candidate density for a conserved quantity on the half line if, at $x^1 = 0$,

$$(T_{+3} + \Theta_{+1} - T_{-3} - \Theta_{-1}) = \partial_0 \Sigma_2. \quad (3.8)$$

Then, provided (3.8) is satisfied, the charge P_2 , given by

$$P_2 = \int_{-\infty}^0 dx^1 (T_{+3} - \Theta_{+1} + T_{-3} - \Theta_{-1}) - \Sigma_2 \quad (3.9)$$

is conserved.

Eq(3.8) is a surprisingly strong condition. Together with the definitions (3.1) and (3.3), it implies that Σ_2 does not exist unless the following two conditions hold at $x^1 = 0$:

$$A_{abc}\mathcal{B}_a + 2B_{ab}\mathcal{B}_{ac} + 2B_{ac}\mathcal{B}_{ab} = 0, \quad (3.10)$$

$$\frac{1}{3}A_{abc}\mathcal{B}_a\mathcal{B}_b\mathcal{B}_c + 2B_{ab}V_a\mathcal{B}_b = 0. \quad (3.11)$$

Both of these involve the boundary term. Comparing (3.10) with (3.4) reveals that the boundary term \mathcal{B} must be equal to

$$\frac{m}{\beta^2} \sum_0^r A_i e^{\frac{\beta}{2}\alpha_i \cdot \phi},$$

apart from an arbitrary additive constant. The second condition, eq(3.11), is nonlinear in the boundary term and therefore provides equations for the constant coefficients A_i in terms of the coefficients in the potential. To analyse these equations, the term in A_{abc} is best eliminated using (3.5), to yield:

$$\frac{1}{24} \sum_{ijk} (B_{ij}C_{ik} + B_{ik}C_{ij}) A_i A_j A_k e_i e_j e_k = \sum_{ij} B_{ij} A_j e_i^2 e_j, \quad (3.12)$$

where

$$e_i = e^{\frac{\beta}{2}\alpha_i \cdot \phi}.$$

Comparing the coefficients of the products of exponentials in (3.12) requires either $A_i = 0$ for all i , or, $A_i^2 = 4$ for all i .

The spin two contribution from the boundary is

$$\Sigma_2 = -\sqrt{2}B_{ab}\partial_0\phi_a\mathcal{B}_b,$$

and $\hat{Q}_2 + \hat{Q}_{-2} - \Sigma_2$ is conserved.

For $a_n^{(1)}$, similar analysis of the spin three and four charges does not lead to any stronger constraints on the coefficients. In [13] it was stated that the spin three charges led to weaker constraints, in the sense that they required the same general form of the boundary condition (1.6), but did not lead by themselves to (1.7). Unfortunately, this conclusion was mistaken. It has since come to light that, except for a_1 , a term had been missed. Once this is taken into account, one is forced to the same conclusion as was reached in the spin two case. For the theories based on $d_n^{(1)}$, spin three is always a possibility; insisting that there be an adaptation of the spin three charge in the presence of a boundary condition requires the coefficients in (1.6) to satisfy (1.8). The spin four charge in the $e_6^{(1)}$ theory similarly constrains the boundary condition in that case.

While the existence of the first few conserved charges is not enough to guarantee integrability, the fact that the rather different calculations for spins two, three and four all lead to the same conditions (1.6), (1.8) seems to be strong evidence for their sufficiency.

4. Classical boundary bound states

With the suggested boundary condition (1.6), the equations of motion for the theory on a half line become

$$\begin{aligned}\partial^2\phi &= -\frac{m^2}{\beta} \sum_0^r n_i \alpha_i e^{\beta \alpha_i \cdot \phi} & x^1 < 0 \\ \partial_1\phi &= -\frac{m}{2\beta} \sum_0^r A_i \alpha_i e^{\frac{\beta}{2} \alpha_i \cdot \phi} & x^1 = 0.\end{aligned}\tag{4.1}$$

With the conventions adopted above, the total conserved energy is given by

$$E = \int_{-\infty}^0 \mathcal{E} dx + \mathcal{B},\tag{4.2}$$

where \mathcal{E} is the usual energy density for Toda field theory. The competition between the two terms when \mathcal{B} is negative permits the existence of boundary bound states.

The coupling constant β can be used to keep track of the scale of the Toda field ϕ , in which case it is appropriate to consider an expansion of the field as a power series in β of the following type:

$$\phi = \sum_{-1}^{\infty} \beta^k \phi^{(k)}.\tag{4.3}$$

Generally, the series starts at $k = -1$ since, with the conventions adopted above, the leading term on the right hand side of the boundary condition is of order $1/\beta$, and may be non-zero. The first two terms of the series satisfy the equations:

$$\begin{aligned}\partial^2\phi^{(-1)} &= -m^2 \sum_0^r n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} & x^1 < 0 \\ \partial_1\phi^{(-1)} &= -\frac{m}{2} \sum_0^r A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} & x^1 = 0,\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}\partial^2\phi^{(0)} &= -m^2 \sum_0^r n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} & x^1 < 0 \\ \partial_1\phi^{(0)} &= -\frac{m}{4} \sum_0^r A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} & x^1 = 0.\end{aligned}\tag{4.5}$$

The linear equations for $\phi^{(0)}$ represent the small coupling limit once the background has been taken into account. Exceptionally, $\phi^{(-1)} = 0$ is a solution to (4.4) when the coefficients

A_i are equal. In view of (1.8), such a situation can arise only for $a_n^{(1)}$. Otherwise, $\phi^{(0)}$ satisfies a linear equation in the background provided by $\phi^{(-1)}$. Since $\phi^{(-1)}$ represents the ‘ground’ state, it is expected to be time-independent and of minimal energy.

In the $a_n^{(1)}$ case, when the coefficients are chosen to be $A_i = A$, $i = 0, \dots, r$, and the ground state is assumed to be $\phi^{(-1)} = 0$, eqs(4.5) reduce to a diagonalisable system whose solution in terms of eigenvectors ρ_a of the mass² matrix may be written as follows:

$$\phi^{(0)} = \sum_{a=1}^r \rho_a (R_a e^{-ip_a x^1} + I_a e^{ip_a x^1}) e^{-i\omega_a x^0}, \quad (4.6)$$

where

$$M^2 \rho_a = m^2 \sum_0^r n_i \alpha_i \otimes \alpha_i \rho_a = m_a^2 \rho_a, \quad \omega_a^2 - p_a^2 = m_a^2,$$

and the reflection factor is given by¹

$$K_a = R_a / I_a = \frac{ip_a + Am_a^2/4m}{ip_a - Am_a^2/4m}, \quad a = 1, \dots, r. \quad (4.7)$$

If $A = 0$, it is clear from (4.7) that $K_a = 1$ and there are no exponentially decaying solutions to the linear system. On the other hand, if $A \neq 0$ the reflection coefficients (4.7) have poles at

$$p_a = -i \frac{Am_a^2}{4m},$$

for which

$$\omega_a^2 = m_a^2 \left(1 - \frac{A^2 m_a^2}{16m^2}\right).$$

Thus, provided $A^2 < 16m^2/m_a^2$ and $A < 0$, the channel labelled a has a bound state, with the corresponding solution to the linear system decaying exponentially away from the boundary as $x^1 \rightarrow -\infty$.

For the case $a_n^{(1)}$, it has been established that $A^2 = 4$, and the masses for the affine Toda theory are known to be

$$m_a = 2m \sin\left(\frac{a\pi}{n+1}\right). \quad (4.8)$$

Hence, with all the $A_i = -2$, there are bound states for each a , with

$$\omega_a^2 = 4m^2 \sin^2\left(\frac{a\pi}{n+1}\right) \left(1 - \sin^2\left(\frac{a\pi}{n+1}\right)\right) = m^2 \sin^2\left(\frac{2a\pi}{n+1}\right). \quad (4.9)$$

Notice that there is a characteristic difference between n odd and n even. In the latter case, the bound-state ‘masses’ are doubly degenerate, matching the degeneracy in the particle states themselves. However, in the former case there is a four-fold degeneracy in the bound-state masses, and $\omega_{(n+1)/2} = 0$.

¹ The notation for the reflection factor is chosen to agree with some earlier references; unfortunately, it also disagrees with others.

5. Quantum boundary bound states

One of the remarkable and intriguing features of the quantum affine Toda field theories based on simply-laced Lie algebras is that the quantum mass spectrum is essentially the same as the classical mass spectrum [2,3]. It is therefore tempting to suppose that a similar miracle might occur for the theories on a half line, in which case the reflection factors corresponding to the special boundary condition $A_i = -2$ (in the case of $a_n^{(1)}$) will contain poles corresponding to the bound-state masses (4.9). Since the S-matrices are known, the reflection factors are strongly constrained (but not uniquely determined) by the various bootstrap relations (2.3).

The simplest case to consider is $a_2^{(1)}$, which contains a conjugate pair of particles with masses given by

$$m_1 = m_2 = \sqrt{3}m. \quad (5.1)$$

The classical reflection factors are given by (4.7), with $A = -2$. It is useful to introduce the block notation (see [2] for details)

$$(x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})}, \quad (5.2)$$

where h is the Coxeter number of the Lie algebra (in this case $h = 3$). In this notation, the classical reflection factor is the same for either particle and may be rewritten as follows:

$$\frac{ip - \frac{3m}{2}}{ip + \frac{3m}{2}} = -(1)(2). \quad (5.3)$$

Note that this expression does satisfy the only classical bootstrap relation (2.4) of the a_2 theory, corresponding to the coupling $11 \rightarrow 2$, with $\theta_{11}^2 = 2i\pi/3$.

In the same notation, the S-matrix elements are given by

$$S_{11}(\theta) = S_{22}(\theta) = \frac{(2)}{(B)(2-B)}; \quad S_{12}(\theta) = S_{11}(i\pi - \theta) = -\frac{(1)}{(1+B)(3-B)}, \quad (5.4)$$

where the parameter B depends on the coupling constant; it has been conjectured to have the form

$$B(\beta) = \frac{\beta^2/2\pi}{1 + \beta^2/4\pi},$$

and checked to one-loop order for all simply-laced affine Toda theories [4]. The boundary condition does not distinguish the two particles and, if the two reflection factors describing

reflection of either particle off the ground state of the boundary are denoted $K_1^0(\theta)$ and $K_2^0(\theta)$, it is expected that

$$K_1^0(\theta) = K_2^0(\theta).$$

In addition, the bootstrap equation [16,10] consistent with the $11 \rightarrow 2$ coupling in the theory

$$K_2^0(\theta) = K_1^0(\theta - i\pi/3)K_1^0(\theta + i\pi/3)S_{11}(2\theta) \quad (5.5)$$

must be satisfied, as must the ‘crossing-unitarity’ relation [10]

$$K_1^0(\theta)K_2^0(\theta - i\pi) = S_{11}(2\theta). \quad (5.6)$$

For the reason mentioned previously, in this case it is sufficient to check the bootstrap properties only; (5.6) is a consequence of the bootstrap property.

On the basis of the discussion in the previous section, the reflection factors are expected to contain a fixed simple pole (at $\theta = i\pi/3$) indicating the existence of the boundary bound state expected in each channel at the mass $\sqrt{3}m/2$. It is also expected that as $\beta \rightarrow 0$ the reflection factors revert to the classical expression (5.3). A ‘minimal’ hypothesis with these properties is:

$$K_1^0(\theta) = K_2^0(\theta) = -\frac{(1)(2 + \frac{B}{2})}{(\frac{B}{2})}. \quad (5.7)$$

Remarkably, this simple ansatz satisfies both the requirements, (2.2) and (2.3), as is easily verified. As $\beta \rightarrow 0$, the β -dependent factors in (5.7) give the rapidity dependent factor (2) in the classical reflection factor (5.3). This expression is not invariant under the transformation $\beta \rightarrow 4\pi/\beta$, the weak-strong coupling symmetry characteristic of the quantum affine Toda theory on the whole line. Rather, as $\beta \rightarrow \infty$, $K_1^0 \rightarrow 1$.

Each channel has a boundary bound state (associated with the pole at $\theta = i\pi/3$), and it is convenient to label these b_1 and b_2 . The boundary bootstrap equation [10] defines the reflection factors for the particles reflecting from the boundary bound states. If, as is being assumed here, there remain sufficiently many charges conserved in the presence of the boundary to ensure that the reflection off the boundary is diagonal, then the equation given by Ghoshal and Zamolodchikov simplifies dramatically. If the scattering of particle a with the boundary state α has a boundary bound state pole at $\theta = iv_{a\alpha}^\beta$, then the reflection factors for the new boundary state are given by

$$K_b^\beta(\theta) = S_{ab}(\theta - iv_{a\alpha}^\beta)S_{ab}(\theta + iv_{a\alpha}^\beta)K_b^\alpha(\theta). \quad (5.8)$$

Thus, for the case in hand, the four possibilities are

$$\begin{aligned} K_1^{b_1} &= S_{11}(\theta + i\pi/3)S_{11}(\theta - i\pi/3)K_1^0(\theta) = S_{12}(\theta)K_1^0 \\ K_2^{b_1} &= S_{12}(\theta + i\pi/3)S_{12}(\theta - i\pi/3)K_2^0(\theta) = S_{11}(\theta)K_2^0 \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} K_1^{b_2} &= S_{12}(\theta + i\pi/3)S_{12}(\theta - i\pi/3)K_1^0(\theta) = S_{11}(\theta)K_1^0 \\ K_2^{b_2} &= S_{11}(\theta + i\pi/3)S_{11}(\theta - i\pi/3)K_2^0(\theta) = S_{12}(\theta)K_2^0. \end{aligned} \quad (5.10)$$

Consider the fixed pole structure of eqs(5.9). Since both S_{12} and K_1^0 have a simple pole at $\theta = i\pi/3$, their product has a double pole; this is not to be interpreted as a new bound state. On the other hand, S_{11} has a simple pole at $\theta = 2i\pi/3$ and K_2^0 has a simple pole at $\theta = i\pi/3$; the first of these does not indicate a new boundary bound state since for that interpretation θ ought to lie in the range $0 \leq \theta \leq i\pi/2$. However, the second pole lies in the correct range and indicates a boundary state of mass $\sqrt{3}m$. This state has all the quantum numbers of particle 1 (the state b_1 has the quantum numbers of particle 2 each multiplied by 1/2), and may therefore be interpreted as a particle 1 state at zero momentum, next to the boundary in its ground state. Establishing the latter relies on the fact that the particle charges and the boundary state charges are related in the quantum field theory via

$$P_s^a \cos(sv_{a\alpha}^\beta) = P_s^\beta - P_s^\alpha. \quad (5.11)$$

Eqs(5.10) have a similar interpretation. Consequently, it is tempting to conjecture that the complete boundary spectrum corresponding to the symmetrical boundary condition (1.6) with $A_1 = A_2 = -2$ consists of a ground state, a pair of boundary states, and a tower of states constructed by gluing zero rapidity particles to either the ground state or to the boundary states b_1 , b_2 . Fring and Köberle [20] have reached a similar pattern of boundary states by examining a particular solution to the e_6 reflection bootstrap equations. They do not, however, link their conjecture to a particular boundary condition.

On the other hand, if $A_1 = A_2 = 2$, the classical reflection data has no boundary bound states and the classical reflection coefficient (5.3) is replaced by its inverse. In this case, a candidate for the reflection factors in the quantum field theory is

$$K_1^0(\theta) = K_2^0(\theta) = \frac{(3 - \frac{B}{2})}{(2)(1 - \frac{B}{2})}. \quad (5.12)$$

This clearly satisfies all the bootstrap conditions and there are no physical strip poles corresponding to boundary bound states. As $\beta \rightarrow \infty$, these reflection factors tend to unity.

In order to generalise (5.7) to other members of the a_n series, it is useful to have some new notation. It is convenient to introduce a pair of new blocks:

$$\langle x \rangle = \frac{(x + \frac{1}{2})}{(x - \frac{1}{2} + \frac{B}{2})}, \quad \langle \widetilde{x} \rangle = \frac{(x - \frac{1}{2})}{(x + \frac{1}{2} - \frac{B}{2})}. \quad (5.13)$$

These are related to the notation $[x]$ introduced in [17] via

$$[x] = \langle x \rangle \langle \widetilde{x} \rangle. \quad (5.14)$$

In terms of (5.14), the quantities $S(2\theta)$ can be manipulated conveniently, since

$$\{x\}(2\theta) = [x/2](\theta)/[h - x/2](\theta), \quad (5.15)$$

where

$$\{x\} = \frac{(x-1)(x+1)}{(x-1+B)(x+1-B)}$$

is the basic building block from which all the S-matrices of simply-laced affine Toda field theories are constructed [2].

In terms of the new blocks, eq(5.7) may be rewritten as

$$K_1^0 = \frac{\langle \frac{1}{2} \rangle}{\langle \frac{5}{2} \rangle} = \frac{\langle \frac{1}{2} \rangle}{\langle h - \frac{1}{2} \rangle},$$

which is in a suitable form to generalise. Following the bootstrap, using it recursively to define all the other reflection factors, leads to the general expression

$$K_a^0 = \frac{\langle a - \frac{1}{2} \rangle}{\langle h - a + \frac{1}{2} \rangle} \frac{\langle a - 1 - \frac{1}{2} \rangle}{\langle h - a + 1 + \frac{1}{2} \rangle} \cdots \frac{\langle \frac{1}{2} \rangle}{\langle h - \frac{1}{2} \rangle} = K_{h-a}^0. \quad (5.16)$$

Moreover,

$$K_a^0 \rightarrow -(a)(h-a), \quad \beta \rightarrow 0$$

and, for each a , $K_a^0 \rightarrow 1$ as $\beta \rightarrow \infty$. The limit $\beta \rightarrow 0$ yields the classical reflection factor (4.7), corresponding to particle a in the field theory based on $a_n^{(1)}$; this also satisfies (2.4).

The generalisation of (5.12) is obtained by replacing $\langle x \rangle$ by $\langle \widetilde{x} \rangle$ in (5.16).

This picture is quite attractive but tentative. It must be emphasised that there is much ambiguity involved in finding solutions to the reflection bootstrap equation (see [17]) and therefore, although it is tempting to use the simplest expressions satisfying the requirements, the solutions suggested here are not guaranteed. Even given a ‘minimality’ assumption, it is quite often the case that the solution is not unique. Further work needs to be done to resolve these ambiguities.

For all other Toda theories, the classical background $\phi^{(-1)} = 0$ is only possible for the relatively simple boundary condition $\mathcal{B} = 0$. This means that the above analysis is irrelevant in general and either the only integrable boundary condition for the other theories in the affine Toda class is the simple one, or one needs to develop a technology to explore the non-trivial static backgrounds. These provide an effective potential for the linearised scattering problem, corresponding to the weak coupling limit, and the derived scattering data provides a selection principle among the known solutions to the bootstrap equations (2.3). Basically, these background solutions are similar to solitons in the complex theory, continued to real coupling. Such solutions are real and inevitably diverge at some value of x^1 . Provided the singularity occurs beyond the boundary, in $x^1 > 0$, there is no cause for alarm.

Obtaining the background field configurations for a given boundary condition is a formidable problem and is certainly unsolved for the general case. A few illustrative examples will be given in the following sections.

6. The $a_2^{(1)}$ theory with an asymmetrical boundary condition

As a first example, consider the case of $a_2^{(1)}$ with an asymmetrical boundary condition. For definiteness, take $A_1 = 2$, $A_2 = A_0 = -2$. For this case, $\phi^{(-1)} = 0$ is not an option and the first task is to solve for the background configuration. The ansatz

$$\phi^{(-1)} = \alpha_1 \rho \tag{6.1}$$

is compatible with the boundary condition and leads to a time-independent² Bullough-Dodd equation ($a_2^{(2)}$ affine Toda):

$$\begin{aligned} \rho'' &= e^{2\rho} - e^{-\rho} & x < 0 \\ \rho' &= -(e^\rho + e^{-\rho/2}) & x = 0. \end{aligned} \tag{6.2}$$

² From now on $x \equiv x^1$

At first sight, (6.2) does not look promising, particularly the boundary condition. The corresponding equations for $\phi^{(0)}$ are (putting $m = 1$):

$$\begin{aligned}
\partial^2 \phi^{(0)} &= - \sum_0^r n_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} & x^1 < 0 \\
&= - \begin{pmatrix} 2e^{2\rho} + e^{-\rho} & 0 \\ 0 & 3e^{-\rho} \end{pmatrix} \phi^{(0)} \\
\partial_1 \phi^{(0)} &= \frac{1}{4} \sum_0^r A_i \alpha_i e^{\frac{1}{2} \alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} & x^1 = 0 \\
&= -\frac{1}{2} \begin{pmatrix} 2e^\rho - e^{-\rho/2} & 0 \\ 0 & -3e^{-\rho/2} \end{pmatrix} \phi^{(0)}.
\end{aligned} \tag{6.3}$$

In the following analysis, it is convenient first to concentrate on the second component of $\phi^{(0)}$, returning to the other later.

Integrating the Bullough-Dodd equation once yields

$$(\rho')^2 = e^{2\rho} + 2e^{-\rho} - 3,$$

and therefore at the boundary $x = 0$

$$(e^\rho + e^{-\rho/2})^2 = e^{2\rho} + 2e^{-\rho} - 3.$$

The latter implies

$$e^{\rho/2} = -1, \ 1/2, \text{ or } \rho \rightarrow \infty.$$

Clearly, the first of these possibilities (which is a double root) is incompatible with the reality of ρ but the second of them is fine; the third implies a singularity at the origin which may be acceptable. In fact, for the above choice of coefficients, the finite solution is the appropriate one. Given this boundary value the boundary condition for the linear approximation around the background is:

$$\partial_1 \phi^{(0)} = \begin{pmatrix} 3/4 & 0 \\ 0 & 3 \end{pmatrix} \phi^{(0)}. \tag{6.4}$$

The relevant solution to Bullough-Dodd, with $\rho \rightarrow 0$ as $x \rightarrow -\infty$, is given by:

$$e^{-\rho} = \frac{1 + 4E + E^2}{1 - 2E + E^2} = 1 + \frac{3/2}{\sinh^2 \sqrt{3}(x - x_0)/2}, \quad E = e^{\sqrt{3}(x - x_0)}, \tag{6.5}$$

where x_0 is to be determined by the boundary condition. It must, however, turn out that $x_0 > 0$ so that the singularity in (6.5) does not occur inside the half line $x < 0$. To check this the boundary condition is matched by

$$e^{-\rho} = 4 = 1 + (3/2)(\coth^2 \sqrt{3}(x_0)/2 - 1),$$

ie

$$\coth^2 \sqrt{3}(x_0)/2 = 3.$$

The positive solution to this must be chosen, given the sign of the slope of ρ near the boundary. Note that the solution (6.5) is a static solution of the type discovered by Aratyn et al [21]. Regarded as a solution to the $a_2^{(1)}$ classical Toda equations, it is of the standard Hirota type but not of the kind originally discussed by Hollowood [14].

Next, consider the equation for $\phi^{(0)}$. We expect to find a solution of the form

$$\phi^{(0)} = e^{-i\omega t} \Phi(x).$$

However, given the form of the background potential, it is convenient to make a change of variable to $z = \sqrt{3}x/2$. Once this has been done, Φ satisfies:

$$\Phi'' = \left(-\lambda^2 + \frac{6}{\sinh^2(z - z_0)} \right) \Phi, \quad (6.6)$$

where it is convenient to set

$$\lambda^2 = (4/3)(\omega^2 - 3) = (4/3)p^2 = 4 \sinh^2(\theta).$$

This is quite a striking result: not only is the ‘potential’ on the right hand side of the equation in the class of exactly solvable ones, it has a coefficient which, in the case of a $1/\cosh^2$ potential, indicates no reflection. Therefore, the equation for Φ is very special, and indeed solvable in elementary terms. The known solution (see [22], for example) has the form

$$\Phi_L = \left(\frac{d}{dz} - 2 \coth(z - z_0) \right) \left(\frac{d}{dz} - \coth(z - z_0) \right) e^{i\lambda z} \quad (6.7)$$

leading to a general solution for Φ of the form

$$\Phi = a\Phi_L + \text{complex conjugate},$$

from which the reflection coefficient can be found. Firstly, as $z \rightarrow -\infty$, $\coth(z - z_0) \rightarrow -1$ and so:

$$\Phi \sim a(i\lambda + 2)(i\lambda + 1)e^{i\lambda z} + a^*(-i\lambda + 2)(-i\lambda + 1)e^{-i\lambda z}. \quad (6.8)$$

Therefore, the reflection coefficient is given by

$$K = \frac{a^*}{a} \frac{(i\lambda - 2)}{(i\lambda + 2)} \frac{(i\lambda - 1)}{(i\lambda + 1)}. \quad (6.9)$$

Next, (6.7) and its derivative at $x = 0$ need to be evaluated in order to fix the relationship between a and a^* . The result is:

$$\begin{aligned} \Phi_L(0) &= (i\lambda)^2 + 3\sqrt{3}(i\lambda) + 8 \\ \Phi'_L(0) &= (i\lambda)^3 + 3\sqrt{3}(i\lambda)^2 + 14(i\lambda) + 12\sqrt{3}. \end{aligned} \quad (6.10)$$

Therefore the boundary condition becomes:

$$a((i\lambda)^3 + \sqrt{3}(i\lambda)^2 - 4(i\lambda) - 4\sqrt{3}) + cc = a(i\lambda + \sqrt{3})((i\lambda)^2 - 4) + cc = 0,$$

from which

$$\frac{a^*}{a} = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}}.$$

Remembering that $\lambda = 2 \sinh \theta \equiv 2s$ leads to

$$K = \frac{is + \sqrt{3}/2}{is - \sqrt{3}/2} \frac{is - 1}{is + 1} \frac{is - 1/2}{is + 1/2},$$

which, in the usual notation (5.2), is just

$$K_1 = \frac{(1/2)(3/2)^2(5/2)}{(1)(2)(3)}. \quad (6.11)$$

Remarkably, this reflection data corresponding to one of the channels implies the same data in the other channel, assuming the ‘classical reflection’ bootstrap equation (2.4) holds. Indeed, the denominator is identical to data obtained above from the symmetrical boundary condition, while the numerator satisfies the bootstrap on its own. Clearly (6.11) can be regarded as the classical limit of a solution to the full bootstrap equation in many ways. The simplest such solution would be

$$K_1 = K_2 = (1/2)(3/2)^2(5/2) \frac{(3 - \frac{B}{2})}{(1 - \frac{B}{2})(2)}, \quad (6.12)$$

but there are many others. For example, if $C(\beta)$ is any function of β , vanishing at $\beta = 0$, then

$$(1/2 + C)(3/2 - C)(3/2 + C)(5/2 - C) \frac{(3 - \frac{B}{2})}{(1 - \frac{B}{2})(2)},$$

is also a solution.

A check on all of this is to calculate directly the reflection data in the other channel and verify that it is indeed the same. The linear approximation in the background potential in the other channel has the form

$$\Phi'' = \left(-\lambda^2 + \frac{4r}{q^2} \right) \Phi,$$

where

$$r = -6E(1 - 6E + 3E^2 + 4E^3 + 3E^4 - 6E^5 + E^6)$$

$$q = 1 + 2E - 6E^2 + 2E^3 + E^4,$$

and, as before,

$$E = e^{\sqrt{3}(x-x_0)} = e^{2(z-z_0)}.$$

The equation is solved by taking Φ to have the form

$$\Phi_L = \frac{p}{q} e^{i\lambda z},$$

where ($i\lambda = 2i \sinh \theta$) and the function p depends on λ and is given up to an overall constant by

$$p = (2 + i\lambda)(1 + i\lambda) - 2(\lambda^2 + 4)(E + E^3) + 6(2 + \lambda^2)E^2 + (2 - i\lambda)(1 - i\lambda)E^4.$$

At $z = 0$ the following boundary condition holds:

$$\Phi'(0) = \frac{\sqrt{3}}{2} \Phi(0).$$

Hence, setting

$$\Phi = a \frac{p}{q} e^{i\lambda z} + cc \tag{6.13}$$

and imposing the boundary condition fixes the ratio a^*/a :

$$\frac{a^*}{a} = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}}.$$

Using the latter, the reflection factor determined by the solution (6.13) is

$$K = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}} \frac{(2 - i\lambda)(1 - i\lambda)}{(2 + i\lambda)(1 + i\lambda)},$$

which is precisely the same as (6.11).

It is striking, and quite surprising, to find that the classical reflection factors obtained by the direct calculations described here do satisfy the simple classical bootstrap equation (2.4); that this is the case provides some support for the approach advocated above. The phenomenon is not confined to the $a_n^{(1)}$ theories, as the next section will show.

7. The $d_5^{(1)}$ theory

As a second example, consider the $d_5^{(1)}$ theory and solve the background equation in a symmetrical situation (ie $A_0 = A_1 = A_4 = A_5 = -2$, $A_2 = A_3 = -2\sqrt{2}$). This choice is expected to contain bound states; again, $\phi^{(-1)} = 0$ is not an option. Hence, there is no choice but to find the background first. There is still a symmetry in the sense of preserving the symmetry of the affine Dynkin diagram for $d_5^{(1)}$, and the ansatz³

$$\phi^{(-1)} = (\alpha_2 + \alpha_3)\rho \tag{7.1}$$

is compatible with the boundary condition and leads to the sinh-Gordon equation for ρ .

$$\begin{aligned} \rho'' &= 2(e^\rho - e^{-\rho}) & x < 0 \\ \rho' &= -2e^{-\rho/2} + \sqrt{2}e^{\rho/2} & x = 0. \end{aligned} \tag{7.2}$$

This looks straightforward but there is a subtlety.

Integrating the first of eqs(7.2), and matching with the second at the boundary gives:

$$2e^{-\rho/2} - \sqrt{2}e^{\rho/2} = 2(e^{-\rho/2} - e^{\rho/2}),$$

which implies that $e^{\rho/2}$ vanishes at the boundary. At first sight, this seems unreasonable. However, a calculation of the energy of this field configuration gives zero; the half line contribution and the boundary term (both infinite) cancel precisely. The relevant solution to the sinh-Gordon equation is

$$e^{\rho/2} = \frac{1 - e^{2x}}{1 + e^{2x}}. \tag{7.3}$$

³ The centre spots of the diagram are labelled ‘2,3’.

The equations for ϕ_0 can now be computed and, in a suitable basis for the roots, the field equation becomes

$$\partial^2 \phi^{(0)} = - \left[e^{-\rho} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} + e^{\rho} \begin{pmatrix} (1-\sqrt{2})^2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & (1+\sqrt{2})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right] \phi^{(0)}. \quad (7.4)$$

Clearly, the only easily solvable equations are those corresponding to the s , \bar{s} particles of mass $\sqrt{2}$, and the particle 2 of mass 2. Since the background potential is singular at $x = 0$, an acceptable solution must vanish at the origin. Using the expression for the background field (7.3), the linearised equation for the s or \bar{s} components takes the form:

$$\partial^2 \phi_s = - \left(2 + \frac{2}{\sinh^2 x} \right) \phi_s,$$

which is again exactly solvable. Setting

$$\phi_s = e^{-i\omega t} \Phi_s,$$

the solution is given by

$$\Phi_s = a(i\lambda - \coth x)e^{i\lambda x} + cc \quad \text{with } \lambda^2 = \omega^2 - 2 = 2 \sinh^2 \theta.$$

Near the origin

$$\coth x = \frac{1}{x}(1 + x^2/3 + O(x^4));$$

therefore, the choice $a^* = -a$ is enough to remove the singularity. Indeed, near the origin $\Phi_s = O(x^2)$ which is exactly right for ensuring the boundary condition is also satisfied at $x = 0$. In other words, the singular behaviour of the background is harmless. Hence,

$$\phi_s = ae^{-i\omega t} [(i\lambda - \coth x)e^{i\lambda x} - cc],$$

from which the reflection data for the classical scattering may be read off, to give

$$K_s = \frac{i\lambda - 1}{i\lambda + 1} = -(2)(6) = K_{\bar{s}}. \quad (7.5)$$

Again, the notation (5.2) is convenient (with $h = 8$). If (7.5) is taken seriously, it implies the presence of a bound state at $\theta = i\pi/4$ with mass $m_s/\sqrt{2}$.

The other diagonal channel has an effective potential

$$-2(e^\rho + e^{-\rho}) = -2\left(2 + \frac{4}{\sinh^2 2x}\right),$$

which is again integrable of the same type apart from the scale of x . Calculating the classical scattering data yields

$$K_2 = \frac{i\lambda - 2}{i\lambda + 2} = -(4)^2.$$

No bound state would be expected in this channel.

On the other hand, it can be assumed that (7.5) provides a starting point for a classical bootstrap calculation; then the bootstrap itself will determine the scattering data for the other particles, provided it is consistent. The d_5 couplings are:

$$ss1 \quad ss3 \quad s\bar{s}2 \quad 112 \quad 123 \quad 332$$

with the appropriate coupling angles given in [2]. Remarkably, (7.5) provides a consistent solution to the classical bootstrap (2.4) based on these couplings for which

$$\begin{aligned} K_s &= K_{\bar{s}} = -(2)(6) \\ K_1 &= \frac{(3)(5)}{(1)(7)} \\ K_2 &= -(4)^2 \\ K_3 &= (1)(3)(5)(7) \end{aligned} \tag{7.6}$$

is the full set of data. Note that the expression calculated above for K_2 is consistent with the bootstrap. A lengthier calculation has been done to check the reflection data in the channels corresponding to particles 1 and 3. This will not be described here for reasons of space, but makes use of the idea that the linearised approximation may be thought of as a particular limit of what would be a double soliton solution in the imaginary coupling theory [23,24]. At first sight it seems unlikely that the scattering data will be diagonal, given the mixing in the linearised equation (7.4). However, not only is the scattering diagonal, the computed reflection data precisely matches the prediction of (7.6)!

There are several possible solutions to the reflection bootstrap equations for which (7.6) are the ‘classical’ limit.

8. The $a_1^{(1)}$ or sinh-Gordon model

As mentioned in section two, this case, in the sense of sine-Gordon, has been studied before. It turns out that in the present context a reasonably complete analysis may be made.

The equation for the static background is (for convenience $\rho = \phi^{(-1)}$)

$$\begin{aligned}\rho'' &= -\sqrt{2} \left(e^{\sqrt{2}\rho} - e^{-\sqrt{2}\rho} \right) & x < 0 \\ \rho' &= -\sqrt{2} \left(\epsilon_1 e^{\rho/\sqrt{2}} - \epsilon_0 e^{-\rho/\sqrt{2}} \right) & x = 0, \quad A_i = 2\epsilon_i\end{aligned}\tag{8.1}$$

from which, on integrating the first equation once, and comparing with the boundary condition, one obtains

$$\begin{aligned}\rho' &= \sqrt{2} \left(e^{\rho/\sqrt{2}} - e^{-\rho/\sqrt{2}} \right) & x < 0 \\ e^{\sqrt{2}\rho} &= \frac{1 + \epsilon_0}{1 + \epsilon_1} & x = 0.\end{aligned}\tag{8.2}$$

Hence, the ground state solution has the form

$$e^{\rho/\sqrt{2}} = \frac{1 + e^{2(x-x_0)}}{1 - e^{2(x-x_0)}},\tag{8.3}$$

with

$$\coth x_0 = \sqrt{\frac{1 + \epsilon_0}{1 + \epsilon_1}}.\tag{8.4}$$

The expression given in (8.4) assumes $\epsilon_0 > \epsilon_1$; if that is not the case, it is necessary to adjust the solution by shifting x_0 by $i\pi/2$.

The linearised wave equation in this background has the form

$$\begin{aligned}\partial^2 \phi^{(0)} &= -4 \left(1 + \frac{2}{\sinh^2 2(x-x_0)} \right) \phi^{(0)} & x < 0 \\ \partial_1 \phi^{(0)} &= -(\epsilon_0 \tanh x_0 + \epsilon_1 \coth x_0) \phi^{(0)} & x = 0.\end{aligned}\tag{8.5}$$

The classical scattering data for this potential is computable in terms of the parameters in the boundary term. It is convenient to set $\phi^{(0)} = e^{-i\omega t} \Phi(z)$, in which case the solution to (8.5) takes the form

$$\Phi(z) = a(i\lambda - \coth(z - z_0))e^{i\lambda z} + cc, \quad \lambda = \sinh \theta,$$

where the ratio of coefficients a^*/a can be computed from the boundary condition. The reflection coefficient may be read off and turns out to be

$$\begin{aligned} K &= \frac{1 - i\lambda}{1 + i\lambda} \frac{(i\lambda)^2 + i\lambda\sqrt{1 + \epsilon_0}\sqrt{1 + \epsilon_1} + (\epsilon_0 + \epsilon_1)/2}{(i\lambda)^2 - i\lambda\sqrt{1 + \epsilon_0}\sqrt{1 + \epsilon_1} + (\epsilon_0 + \epsilon_1)/2} \\ &= -(1)^2 [(1 + a_0 + a_1)(1 - a_0 + a_1)(1 + a_0 - a_1)(1 - a_0 - a_1)]^{-1}, \end{aligned} \quad (8.6)$$

where in the last step it has been convenient to set

$$\epsilon_i = \cos a_i \pi, \quad |a_i| \leq 1, \quad i = 0, 1.$$

To extend beyond the restriction on the a_i , it is necessary to continue the formula (8.6) by making the substitution $a_i \rightarrow a_i + 2$.

The classical result (8.6) is quite elegant and in fact remarkably similar to the expression for the quantum reflection factor suggested by Ghoshal [11] in the case of the scattering of the lightest sine-Gordon breather state from the boundary⁴.

As a final check on the stability of the background, it is interesting to examine the energy as a functional of the field $\phi^{(-1)} \equiv \rho$. Including the boundary contribution, the energy is given by:

$$E = \int_{-\infty}^0 dx \left(\frac{(\rho')^2}{2} + (e^{\sqrt{2}\rho} + e^{-\sqrt{2}\rho} - 2) \right) + A_1 e^{\rho_0/\sqrt{2}} + A_0 e^{-\rho_0/\sqrt{2}}. \quad (8.7)$$

Using the Bogomolny argument, this may be rewritten, replacing the integrand by

$$\frac{1}{2} \left(\rho' - \sqrt{2}(e^{\rho/\sqrt{2}} - e^{-\rho/\sqrt{2}}) \right)^2 + \sqrt{2}\rho'(e^{\rho/\sqrt{2}} - e^{-\rho/\sqrt{2}}),$$

to obtain

$$E \geq -4 + (A_0 + 2)e^{-\rho_0/\sqrt{2}} + (A_1 + 2)e^{\rho_0/\sqrt{2}}. \quad (8.8)$$

From this it is clear that the energy is definitely bounded below provided both A_0 and A_1 are at least -2 .

⁴ To make the comparison, a suitable ‘classical’ limit of Ghoshal’s formula must be taken after analytic continuation in β .

9. Comments

This is work in progress and there remains much to do. The principal question is: what, if any, boundary conditions are compatible with integrability? From a classical point of view, there are strong constraints on the permitted boundary conditions for affine Toda field theory. From the point of view of the quantum field theory, the situation is less clear and the story is far from complete. On the assumption that a form of integrability survives, such that scattering and reflection from the boundary remain elastic and factorisable, it is possible to make a variety of conjectures for the reflection factors. It is not clear to what extent any of the solutions listed in [16,17,20], or [13], can be said to follow from a particular boundary condition. There are some conjectures but in the end a proper formulation of the theory, via perturbation theory or otherwise, will be needed to decide the issues. On the other hand, the search for the ground state corresponding to a particular choice of boundary condition, and the study of this linearised classical problem, are of interest in themselves. A proper understanding of the classical problem would appear to be a necessary prerequisite to a formulation of perturbation theory in anything other than a situation with a trivial condition at the boundary. The discussion of a system with two boundaries would be interesting from the point of view of finite size effects, and possibly for string theory, particularly in view of the extra states in the spectrum of the theory once boundaries are included. Finally, the boundary conditions considered here have been assumed to be impenetrable. However, there are other possibilities [25] involving the inclusion of internal boundaries, impurities or defects, which allow transmission as well as reflection; these are interesting in their own right, but have been deliberately excluded from the work discussed here.

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